

# Minimal Investment Risk of Portfolio Optimization Problem with Budget and Investment Concentration Constraints

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In the present paper, the minimal investment risk for a portfolio optimization problem with imposed budget and investment concentration constraints is considered using replica analysis. Since the minimal investment risk is influenced by the investment concentration constraint (as well as the budget constraint), it is intuitive that the minimal investment risk for the problem with an investment concentration constraint be larger than that without the constraint (that is, with only the budget constraint). Moreover, a numerical experiment shows the effectiveness of our proposed analysis.

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## I. INTRODUCTION

Portfolio optimization problems constitute one of the most important themes in the research field of mathematical finance and first appeared in the theory of diversification investment introduced by Markowitz in 1952. Analytical procedures for solving portfolio optimization problems to accurately implement asset management so as to disperse the risk by diversifying investment into several assets have become well known [1–4]. Over the next few decades, several issues related to portfolio optimization problems have been addressed, and recently several models and the behavior of the minimal investment risk in portfolio optimization problems have been thoroughly examined using analytical approaches which have been developed and improved through multidisciplinary collaboration [5–9]. Taking advantage of such earlier works, we can find the behavior of the optimal solution for an optimization problem with respect to stochastic phenomena; for instance, we can assess the learning performance of perceptron learning and can evaluate the performance of the decoding algorithm of low-density parity-check code and code division multiple access using an analytical approach developed in statistical mechanical informatics [10–12]. Obviously, since portfolio optimization problems are formulated as stochastic problems, that is, modeled within the framework of probabilistic inference, in order to investigate the behavior of the optimal portfolio, it is useful to adopt an analytical method whose effectiveness has been verified in several research fields. For instance, Ciliberti et al. analyzed the minimal investment risk under an absolute deviation model and an expected shortfall model using replica analysis in the absolute zero temperature limit [5]. Kondor et al. examined quantitatively the noise sensitivity of the optimal portfolio for several risk functions [6]. Pafka et al. discussed the relation between predicted risk, realized risk, and true risk in detail via a scenario ratio (between the number of scenarios and the number of assets) [7]. Shinzato derived the

statistics minimal investment risk and investment concentration and showed that the minimal investment risk is attained and the investment concentration constraint is satisfied (e.g., by a portfolio) and that these two statistics have the self-averaging property which is frequently used in statistical mechanical informatics analysis [8]. In addition, Furthermore, Shinzato et al. developed a faster algorithm for solving portfolio optimization problems using a belief propagation method [9].

However, this previous work mainly investigated the minimal investment risk of the portfolio optimization problem with only one constraint, for instance, a budget constraint. Therefore, we need to analyze in detail the minimal investment risk of a portfolio optimization problem with several constraints, where these constraints correspond to several policies of the stakeholder, so as to handle more practical situations. As the first step of analyzing portfolio optimization problems with several constraints, noting the risk minimization problem, which makes it possible to characterize the investment strategy of a risk-averse investor, our purpose in this study is to develop a novel approach for solving a portfolio optimization problem with two representative constraints, a budget constraint and a constraint on investment concentration, as well as to assess theoretically the minimal investment risk and investment concentration using replica analysis. In these previous works, the minimal investment risk and the investment concentration of the portfolio optimization problem with only a budget constraint are discussed in detail. However, we can also consider an investment concentration constraint, since we assess quantitatively how the minimal investment risk is influenced by the investment concentration, which can be expected to lead to new findings regarding risk-averse investors.

This paper is organized as follows: In the next section, we prepare one of the most analyzed models with respect to portfolio optimization problems, the mean-variance model, by formulating it with two constraints, a budget constraint and an investment concentration constraint. In section III, replica analysis is applied to the portfolio optimization problem with these two constraints, fol-

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lowed by an analytical procedure used in previous work. In section IV, in order to verify the effectiveness of our proposed approach, we compare the results derived using the proposed method with those from numerical simulation and those obtained using the standard approach in operations research. The final section is devoted to summarizing the paper and discussing possible future work.

## II. MEAN-VARIANCE MODEL WITH TWO CONSTRAINTS

This paper considers optimally diversified investment in  $N$  assets in an investment market with no restrictions on short selling. Herein,  $w_i$  is the amount of asset  $i$  ( $= 1, \dots, N$ ) in the portfolio and the full portfolio of  $N$  assets is denoted  $\vec{w} = \{w_1, \dots, w_N\}^T \in \mathbf{R}^N$ , where  $T$  indicates the transpose of a vector or matrix.  $x_{i\mu}$  is the return rate of asset  $i$  under scenario  $\mu$  ( $= 1, \dots, p$ ). For simplicity of our discussion, similar to in the previous work, it is assumed that each return rate  $x_{i\mu}$  is independently and identically normally distributed with mean 0 and variance 1 [8]. Under this assumption, given the  $p$  return rate vectors  $\vec{x}_1, \dots, \vec{x}_p \in \mathbf{R}^N$ ,  $\vec{x}_\mu = \{x_{1\mu}, \dots, x_{N\mu}\}^T \in \mathbf{R}^N$ , written in matrix notation as the return rate matrix  $X = \left\{ \frac{x_{i\mu}}{\sqrt{N}} \right\} \in \mathbf{R}^{N \times p}$ , in the mean-variance model, the investment risk  $\mathcal{H}(\vec{w}|X)$  of portfolio  $\vec{w}$  is defined as follows:

$$\mathcal{H}(\vec{w}|X) = \frac{1}{2} \sum_{\mu=1}^p \left( \frac{1}{\sqrt{N}} \sum_{i=1}^N w_i x_{i\mu} \right)^2. \quad (1)$$

Note that the necessary and sufficient condition for the optimal portfolio for portfolio optimization problem to be uniquely determined given in [8] is that  $J = XX^T \in \mathbf{R}^{N \times N}$  be a non-singular matrix, that is, the rank of matrix  $J$  be  $N$  or simply  $p > N$ . However, since  $J$  does not always need to be a regular matrix to guarantee a unique optimal portfolio for a portfolio optimization problem with several constraints, as in the present work, we do not adopt the regular matrix assumption here. Moreover, coefficient  $1/\sqrt{N}$  guarantees that the statistics defined below are significant, since the expectation of the return rate is assumed to be 0 and the expected return of portfolio  $\vec{w}$  is regarded as 0, we omit terms of expected return. Furthermore, as one of the constraints, the budget constraint is as follows:

$$\sum_{i=1}^N w_i = N. \quad (2)$$

Note that this constraint differs from the budget constraint in the standard context of operations research, since the investment ratios between asset  $i$  and asset  $j$  in optimal portfolios derived from both budget constraints are consistent with each other and we can rescale the budget constraint satisfied with those statistics discussed

hereafter for characterizing the investment style of the optimal portfolio, we employ Eq. (2). For further details about the budget constraint, please refer to [8].

If the portfolio which minimizes the investment risk function in Eq. (1) under the budget constraint in Eq. (2) is represented as  $\vec{w}^* = \{w_1^*, \dots, w_N^*\}^T = \arg\min_{\vec{w}} \mathcal{H}(\vec{w}|X)$ , from a finding in [8], it is guaranteed that the minimal investment risk per asset  $\varepsilon$  is attained, its investment concentration  $q_w$  satisfies the investment risk constraint, and these two statistics have the self-averaging property as  $N$  and  $p$  approach infinity while keeping  $p/N$  bounded [8, 10–12]. Specifically, since  $\mathcal{H}(\vec{w}^*|X) = E_X[\mathcal{H}(\vec{w}^*|X)]$  and  $\sum_{i=1}^N (w_i^*)^2 = \sum_{i=1}^N E_X[(w_i^*)^2]$ , where  $E_X[f(X)]$  means the expectation of  $f(X)$  with respect to return rate matrix  $X$ , the minimal investment risk per asset and the investment concentration are given analytically as follows:

$$\begin{aligned} \varepsilon &= \frac{1}{N} \mathcal{H}(\vec{w}^*|X) \\ &= \begin{cases} \frac{\alpha-1}{2} & \alpha > 1 \\ 0 & \text{otherwise} \end{cases}, \end{aligned} \quad (3)$$

$$\begin{aligned} q_w &= \frac{1}{N} \sum_{i=1}^N (w_i^*)^2 \\ &= \begin{cases} \frac{\alpha}{\alpha-1} & \alpha > 1 \\ \infty & \text{otherwise} \end{cases}, \end{aligned} \quad (4)$$

in which  $\alpha = p/N \sim O(1)$  is the scenario ratio.

Therefore, as is intuitive, the optimal portfolio  $\vec{w}^*$  that minimizes investment risk  $\mathcal{H}(\vec{w}|X)$  in Eq. (1), which is defined for a given return rate matrix  $X$ , depends on return rate matrix  $X$ . However, since the minimal investment risk is attained, the investment concentration satisfies its constraint, and both have the self-averaging property, one can accurately assess their values using replica analysis more easily than using numerical simulations. In contrast, in operations research, based on the analytical approach derived from the principle of expected utility maximization, one solves for the portfolio which minimizes the expected investment risk  $E_X[\mathcal{H}(\vec{w}|X)]$ ,  $\vec{w}^{*OR} = \{w_1^{*OR}, \dots, w_N^{*OR}\}^T = \arg\min_{\vec{w}} E_X[\mathcal{H}(\vec{w}|X)]$ , and then the minimal expected investment risk per asset  $\varepsilon^{OR}$  and its investment concentration  $q_w^{OR}$  can be estimated briefly as follows [8]:

$$\begin{aligned} \varepsilon^{OR} &= \frac{1}{N} E_X[\mathcal{H}(\vec{w}^{*OR}|X)] \\ &= \frac{\alpha}{2}, \quad \alpha > 0, \end{aligned} \quad (5)$$

$$\begin{aligned} q_w^{OR} &= \frac{1}{N} \sum_{i=1}^N (w_i^{*OR})^2 \\ &= 1, \quad \alpha > 0. \end{aligned} \quad (6)$$

From Eq. (3) and Eq. (5),  $\varepsilon < \varepsilon^{OR}$  is obtained, under appropriately realistic conditions, the portfolio which minimizes the expected investment risk  $\vec{w}^{*OR}$  derived using

the standard operations research approach cannot minimize  $\mathcal{H}(\vec{w}|X)$  characterized by an arbitrary return rate matrix  $X$ . Therefore, it is necessary to estimate properly the optimal investment strategy that can minimize the investment risk with respect to a return rate matrix. Finally, from the previous argument, the analytical approach widely used in operations research is regarded as an annealed disordered system approach, which has already been shown in previous interdisciplinary works to be impractical due to the impossibility of analyzing the quenched disordered system and of evaluating the expectation of the optimal objective function [8–12]. That is, the approach of annealed disordered systems can provide little valuable knowledge regarding investing to investors.

In contrast, it has already been shown in the interdisciplinary research [8–12] that the quenched disordered system approach can easily be used to analyze a quenched disordered system and evaluate the expectation of an optimal objective function and we can evaluate the inherent investment risk of an investment system using the quenched disordered system approach and obtain a variety of valuable knowledge and several optimal investment strategies for investors. To achieve the above, as a first step toward precisely analyzing a given investment system which satisfies the definition of a quenched disordered system, we must investigate the potential risk of an investment system.

In a previous study [8], the minimal investment risk and its investment concentration of a portfolio optimization problem with only a budget constraint were analyzed. Therefore, in the present paper, we consider the two-constraint case; namely, the following novel constraint is added:

$$\frac{1}{N} \sum_{i=1}^N w_i^2 = \tau, \quad (7)$$

where  $\tau$  is a scalar constant. This constraint implies that the investment concentration defined in Eq. (4) is held constant. Thus, the portfolio optimization problem discussed in the previous work is extended to the optimization problem of finding  $\vec{w}$  that minimizes the investment risk in Eq. (1) under the budget constraint in Eq. (2) and the investment concentration constraint in Eq. (7).

Let us describe the investment concentration before discussing this portfolio optimization problem. It is easily understood through a comparison between two investment strategies: (concentrated investment strategy: CIS) the investor invests in asset 1 only, that is,  $\vec{w}^{\text{CIS}} = \{N, 0, \dots, 0\}^T \in \mathbf{R}^N$ ; and (equipartition investment strategy: EIS) the investor invests equally in  $N$  assets, that is,  $\vec{w}^{\text{EIS}} = \{1, \dots, 1\}^T \in \mathbf{R}^N$ . Then  $q_w^{\text{CIS}} (= N) > q_w^{\text{EIS}} (= 1)$ . In general, the larger the investment concentration is, the fewer the assets the investor tends to invest in. That is, under the investment concentration and budget constraints, we should determine the optimal portfolio among the set of portfolios whose investment concentrations are  $\tau$ , which is equivalent to

modeling the case of preventing overconcentration in investing.

Eq. (2) and Eq. (7) imply

$$\begin{aligned} \tau - 1 &= \frac{1}{N} \sum_{i=1}^N w_i^2 - \left( \frac{1}{N} \sum_{i=1}^N w_i \right)^2 \\ &= \frac{1}{N} \sum_{i=1}^N \left( w_i - \frac{1}{N} \sum_{i=1}^N w_i \right)^2. \end{aligned} \quad (8)$$

Thus, the constant  $\tau$  in Eq. (7) is at least 1. Moreover, we apply the budget constraint in Eq. (2) rather than  $\sum_{i=1}^N w_i = 1$  as widely used in operations research, because the latter constraint cannot be satisfied simultaneously with the relation Eq. (8), making the constraints hard to interpret in the context of statistics. As in previous studies [5, 7, 8] we normalize the return rate to mean 0, as described previously. Because of this, the sum of the expected returns of the assets of the portfolio is also 0 and cannot be used as a constraint. Therefore, we instead constrain the investment concentration.

Our aim in this work is to compare the minimal investment risk for the budget constraint in Eq. (2) with that with both the budget constraint in Eq. (2) and the investment concentration constraint in Eq. (7). A further aim is to compare the minimal investment risk under the influence of added constraints.

### III. REPLICA ANALYSIS

In a similar way to that used in the previous work [8], we also employ replica analysis as developed in statistical mechanical informatics so as to analyze the minimal investment risk in the portfolio optimization problem with constraints Eq. (2) and Eq. (7). If the investment risk in the market  $\mathcal{H}(\vec{w}|X)$  is regarded as the Hamiltonian of the canonical ensemble, then the partition function of inverse temperature  $\beta$  is defined as follows:

$$Z(X) = \int_{-\infty}^{\infty} d\vec{w} \delta(\vec{w}^T \vec{e} - N) \delta(\vec{w}^T \vec{w} - N\tau) e^{-\beta \mathcal{H}(\vec{w}|X)}, \quad (9)$$

where  $\vec{e} = \{1, \dots, 1\}^T \in \mathbf{R}^N$  and constraints Eq. (2) and Eq. (7) are handled using the delta function.

For a statistical mechanical informatics scenario, we need to find the Helmholtz free energy per asset (or free entropy per asset [13]) in a quenched disordered system in order to calculate the minimal investment risk per asset. For the present study, similar to in previous work [8], since it is difficult to directly implement configuration averaging of the logarithm of the partition function, we analyze this problem using configuration averaging of a power of the partition function and the replica trick [14]. Here we define the following order parameters:

$$q_{wab} = \frac{1}{N} \sum_{i=1}^N w_{ia} w_{ib}, \quad (a, b = 1, \dots, n). \quad (10)$$

Moreover, conjugate order parameters  $\tilde{q}_{wab}$  are prepared and we assume the following replica symmetry solution:

$$q_{wab} = \begin{cases} \chi_w + q_w & a = b \\ q_w & a \neq b \end{cases}, \quad (11)$$

$$\tilde{q}_{wab} = \begin{cases} \tilde{\chi}_w - \tilde{q}_w & a = b \\ -\tilde{q}_w & a \neq b \end{cases}, \quad (12)$$

$$k_a = k, \quad (13)$$

$$\theta_a = \theta, \quad (14)$$

where  $k_a$  and  $\theta_a$  are the conjugate variables of the constraint conditions in Eq. (2) and Eq. (7), respectively. Next, we define

$$\begin{aligned} \phi &= \lim_{N \rightarrow \infty} \frac{1}{N} E_X [\log Z(X)] \\ &= \text{Extr}_{k, \theta, \chi_w, q_w, \tilde{\chi}_w, \tilde{q}_w} \left\{ -k - \frac{\tau\theta}{2} + \frac{(\chi_w + q_w)(\tilde{\chi}_w - \tilde{q}_w)}{2} \right. \\ &\quad + \frac{q_w \tilde{q}_w}{2} - \frac{1}{2} \log(\tilde{\chi}_w - \theta) + \frac{\tilde{q}_w + k^2}{2(\tilde{\chi}_w - \theta)} \\ &\quad \left. - \frac{\alpha}{2} \log(1 + \beta\chi_w) - \frac{\alpha\beta q_w}{2(1 + \beta\chi_w)} \right\}, \end{aligned} \quad (15)$$

where  $\text{Extr}_m g(m)$  means the extremum of  $g(m)$  with respect to  $m$ . Now, the saddle point equation of the parameters is obtained as follows:

$$k = \tilde{\chi}_w - \theta, \quad (16)$$

$$\theta = \tilde{\chi}_w - \frac{1}{\chi_w}, \quad (17)$$

$$\chi_w = \tau - q_w, \quad (18)$$

$$q_w = \chi_w^2 \tilde{q}_w + 1, \quad (19)$$

$$\tilde{\chi}_w = \frac{\alpha\beta}{1 + \beta\chi_w}, \quad (20)$$

$$\tilde{q}_w = \frac{\tilde{\chi}_w^2}{\alpha} q_w. \quad (21)$$

Solving these in the limit of large  $\beta$ , we have

$$\chi_w = \begin{cases} \frac{1}{\beta(\sqrt{\frac{\alpha\tau}{\tau-1}} - 1)} & 1 - \frac{1}{\tau} \leq \alpha \\ \tau - \frac{1}{1-\alpha} & \text{otherwise} \end{cases}, \quad (22)$$

$$q_w = \begin{cases} \tau - \frac{1}{\beta(\sqrt{\frac{\alpha\tau}{\tau-1}} - 1)} & 1 - \frac{1}{\tau} \leq \alpha \\ \frac{1}{1-\alpha} & \text{otherwise} \end{cases}. \quad (23)$$

For both cases of  $\alpha$ , from Eq. (4), Eq. (10), and Eq. (11) with  $a = b$ , the investment concentration of the optimal portfolio is

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (w_i^*)^2 &= \chi_w + q_w \\ &= \tau, \end{aligned} \quad (24)$$

which also satisfies Eq. (7); alternatively, Eq. (24) can be derived directly from Eq. (18). Moreover, following the statistical mechanical informatics literature, if the

minimal investment risk per asset  $\varepsilon$  is determined from  $\varepsilon = \lim_{\beta \rightarrow \infty} \left\{ -\frac{\partial \phi}{\partial \beta} \right\}$ , then

$$\begin{aligned} \varepsilon &= \lim_{\beta \rightarrow \infty} \left\{ \frac{\alpha\chi_w}{2(1 + \beta\chi_w)} + \frac{\alpha q_w}{2(1 + \beta\chi_w)^2} \right\} \\ &= \begin{cases} \frac{\alpha\tau + \tau - 1 - 2\sqrt{\alpha\tau(\tau-1)}}{2} & 1 - \frac{1}{\tau} \leq \alpha \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \quad (25)$$

If  $1 - \frac{1}{\tau} \leq \alpha$ , since the numerator in Eq. (25) can be factored as  $\alpha\tau + \tau - 1 - 2\sqrt{\alpha\tau(\tau-1)} = (\sqrt{\alpha\tau} - \sqrt{\tau-1})^2$ , it is clear that the minimal investment risk per asset is always non-negative. Furthermore, if one shifts  $\varepsilon$  to

$$\varepsilon = \frac{\alpha - 1}{2} + \frac{(\sqrt{\alpha(\tau-1)} - \sqrt{\tau})^2}{2}, \quad (26)$$

then  $\varepsilon$  is guaranteed to be greater than Eq. (3), the minimal investment risk for the portfolio optimization problem with Eq. (7). On the other hand, in the case of  $\sqrt{\alpha(\tau-1)} - \sqrt{\tau} = 0$ ,  $\tau = \alpha/(\alpha-1)$ , as in Eq. (7), and the problem can be regarded as the portfolio optimization problem with a budget constraint.

Using the standard operations research approach, in a similar way to Eq. (5) and Eq. (6), the minimal expected investment risk per asset of the solution to the portfolio optimization problem with both constraints, Eq. (2) and Eq. (7),  $\varepsilon^{\text{OR}}$  and its investment concentration  $q_w^{\text{OR}}$  are easily derived as follows:

$$\varepsilon^{\text{OR}} = \frac{\alpha\tau}{2}, \quad (27)$$

$$q_w^{\text{OR}} = \tau. \quad (28)$$

If  $\alpha > 1 - \frac{1}{\tau}$ ,  $\tau \geq 1$ , then  $\tau - 1 - 2\sqrt{\alpha\tau(\tau-1)} < 0$  holds; if  $0 < \alpha \leq 1 - \frac{1}{\tau}$ ,  $\tau \geq 1$ , then  $\alpha\tau > 0$  also holds, that is, the minimal investment risk per asset  $\varepsilon$  is smaller than the minimal expected investment risk per asset  $\varepsilon^{\text{OR}}$ ,

$$\varepsilon < \varepsilon^{\text{OR}}. \quad (29)$$

Further, from Eq. (6),  $\tau = 1$  is confirmed, so that Eq. (27) and Eq. (28) agree with the analytical findings, Eq. (5) and Eq. (6), in the case of a portfolio optimization problem with a budget constraint only.

Moreover, with respect to the result derived from our proposed approach, instead of the investment concentration constraint, Eq. (7),

$$\tau_0 \leq \frac{1}{N} \sum_{i=1}^N w_i^2, \quad (30)$$

that is, one determines the portfolio which minimizes the investment risk of Eq. (1) in the set of portfolios whose investment concentrations are larger than constant  $\tau_0$ . If  $\alpha > 1$ , then the minimal investment risk per asset  $\varepsilon(\tau_0)$

and its investment concentration  $q_w(\tau_0)$  are as follows:

$$\begin{aligned} \varepsilon(\tau_0) &= \min_{\tau_0 \leq \tau} \left[ \frac{\alpha\tau + \tau - 1 - 2\sqrt{\alpha\tau(\tau-1)}}{2} \right] \\ &= \begin{cases} \frac{\alpha-1}{2} & 1 \leq \tau_0 < \frac{\alpha}{\alpha-1} \\ \frac{\alpha\tau_0 + \tau_0 - 1 - 2\sqrt{\alpha\tau_0(\tau_0-1)}}{2} & \frac{\alpha}{\alpha-1} \leq \tau_0 \end{cases}, \end{aligned} \quad (31)$$

$$q_w(\tau_0) = \begin{cases} \frac{\alpha}{\tau_0} & 1 \leq \tau_0 < \frac{\alpha}{\alpha-1} \\ \frac{\alpha}{\alpha-1} & \frac{\alpha}{\alpha-1} \leq \tau_0 \end{cases}. \quad (32)$$

If  $0 < \alpha \leq 1$ , we have

$$\varepsilon(\tau_0) = 0 \quad 1 \leq \tau_0, \quad (33)$$

$$q_w(\tau_0) \geq \frac{1}{1-\alpha} \quad 1 \leq \tau_0. \quad (34)$$

In a similar way, for the case of

$$\tau_0 \geq \frac{1}{N} \sum_{i=1}^N w_i^2, \quad (35)$$

then if  $\alpha > 1$ ,  $\varepsilon(\tau_0)$  and  $q_w(\tau_0)$  are

$$\begin{aligned} \varepsilon(\tau_0) &= \min_{\tau_0 \geq \tau} \left[ \frac{\alpha\tau + \tau - 1 - 2\sqrt{\alpha\tau(\tau-1)}}{2} \right] \\ &= \begin{cases} \frac{\alpha\tau_0 + \tau_0 - 1 - 2\sqrt{\alpha\tau_0(\tau_0-1)}}{2} & 1 \leq \tau_0 < \frac{\alpha}{\alpha-1} \\ \frac{\alpha-1}{2} & \frac{\alpha}{\alpha-1} \leq \tau_0 \end{cases}, \end{aligned} \quad (36)$$

$$q_w(\tau_0) = \begin{cases} \tau_0 & 1 \leq \tau_0 < \frac{\alpha}{\alpha-1} \\ \frac{\alpha}{\alpha-1} & \frac{\alpha}{\alpha-1} \leq \tau_0 \end{cases}, \quad (37)$$

whereas if  $0 < \alpha \leq 1$ , they are given by

$$\varepsilon(\tau_0) = \begin{cases} \frac{\alpha\tau_0 + \tau_0 - 1 - 2\sqrt{\alpha\tau_0(\tau_0-1)}}{2} & 1 \leq \tau_0 < \frac{1}{1-\alpha} \\ 0 & \frac{1}{1-\alpha} \leq \tau_0 \end{cases}, \quad (38)$$

$$q_w(\tau_0) = \begin{cases} \tau_0 & 1 \leq \tau_0 < \frac{1}{1-\alpha} \\ \frac{1}{1-\alpha} + c & \frac{1}{1-\alpha} \leq \tau_0 \end{cases}, \quad (39)$$

for any  $c \geq 0$ . On the other hand, if considered in the context of operations research, with respect to both optimization problems, that is, those with the investment concentration constraint modified to Eq. (30) or Eq. (35), the minimal expected investment risk per asset  $\varepsilon^{\text{OR}}(\tau_0)$  and its investment concentration  $q_w^{\text{OR}}(\tau_0)$  are easily determined as follows:

$$\varepsilon^{\text{OR}}(\tau_0) = \frac{\alpha\tau_0}{2}, \quad (40)$$

$$q_w^{\text{OR}}(\tau_0) = \tau_0, \quad (41)$$

for which  $\varepsilon(\tau_0) < \varepsilon^{\text{OR}}(\tau_0)$  holds. Namely, we can verify intuitively the findings derived from replica analysis for these two problems.

#### IV. NUMERICAL EXPERIMENT

In order to verify our proposed approach based on the assumption of a replica symmetry solution and the thermodynamic limit of the number of assets or scenarios, we should assess numerically the portfolio which minimizes the investment risk for the optimization problem with the constraints Eq. (2) and Eq. (7) using the Lagrange undetermined multipliers method, which can solve for the optimal solution without these assumptions and compare the analytical results via numerical simulation. The Lagrange function is defined as follows:

$$L(\vec{w}, k, \theta) = \frac{1}{2} \vec{w}^T J \vec{w} + k(N - \vec{w}^T \vec{e}) - \frac{\theta}{2} (\vec{w}^T \vec{w} - N\tau), \quad (42)$$

where the  $i, j$  component of variance-covariance matrix  $J (= XX^T) = \{J_{ij}\} \in \mathbf{R}^{N \times N}$  is

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^p x_{i\mu} x_{j\mu}. \quad (43)$$

We optimize Lagrange function  $L(\vec{w}, k, \theta)$  by using the following algorithm based on the steepest descent method. Given a return rate matrix  $X = \left\{ \frac{x_{i\mu}}{\sqrt{N}} \right\} \in \mathbf{R}^{N \times p}$ , the initial states of portfolio  $\vec{w}$  and two conjugate variables  $k, \theta$  are adequately initialized, for instance,  $\vec{w}^0 = \vec{e}$  and  $k^0 = \theta^0 = 1$ . At iteration step  $s$ ,  $\vec{w}^s, k^s, \theta^s$  are updated as follows:

$$\vec{w}^{s+1} = \vec{w}^s - \eta_w \frac{\partial L(\vec{w}^s, k^s, \theta^s)}{\partial \vec{w}^s}, \quad (44)$$

$$k^{s+1} = k^s + \eta_k \frac{\partial L(\vec{w}^s, k^s, \theta^s)}{\partial k^s}, \quad (45)$$

$$\theta^{s+1} = \theta^s + \eta_\theta \frac{\partial L(\vec{w}^s, k^s, \theta^s)}{\partial \theta^s}, \quad (46)$$

where learning steps  $\eta_w, \eta_k, \eta_\theta$  are infinitesimal positive numbers, set as  $\eta_w = \eta_k = \eta_\theta = 10^{-1}$  in the present experiment. Moreover, when the difference  $\Delta = \sum_{i=1}^N |w_i^s - w_i^{s+1}| + |k^s - k^{s+1}| + |\theta^s - \theta^{s+1}|$  is less than  $10^{-4}$ , then numerical solution  $\vec{w}^s$  is regarded as an approximation solution of the optimal portfolio  $\vec{w}^* = \arg \min_{\vec{w}} \mathcal{H}(\vec{w}|X)$ , and is used to estimate the minimal investment risk per asset  $\varepsilon$ .

The number of assets in a numerical simulation  $N$  is set as  $N = 500$  and the scenario ratio  $\alpha = p/N$  is set as  $\alpha = 2$ . Furthermore, we set return rate of each asset  $x_{i\mu}$  as independently and identically distributed following the standard normal distribution  $N(0, 1)$ , and 100 return rate matrices  $X^1, \dots, X^{100}$  are prepared as sample sets. We assess the average of minimal investment risk per asset using the optimal portfolio of each sample,  $\vec{w}^{*,1}, \dots, \vec{w}^{*,100}$ , where  $\vec{w}^{*,c} = \arg \min_{\vec{w}} \mathcal{H}(\vec{w}|X^c)$  is obtained using the above-described Lagrange undetermined multipliers algorithm. In Fig. 1, the analytical results derived by replica analysis (Eq. (25)) are shown with the analytical results obtained by the standard operations research approach (Eq. (27)) and the practical results

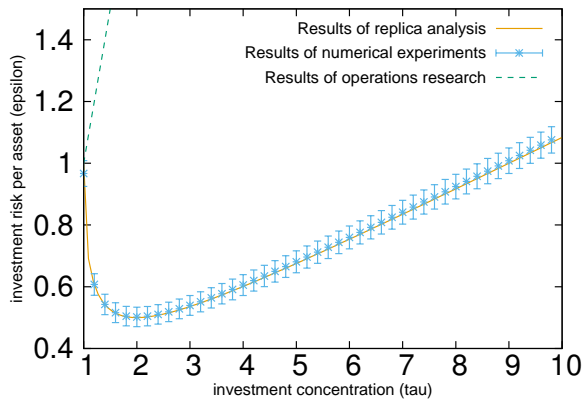


FIG. 1. Minimal investment risk per asset  $\varepsilon$  at  $\alpha = p/N = 2$  results from replica analysis (orange line), numerical simulation (sky-blue asterisks with error bars), and operations research approach (green dashed line) versus investment concentration  $\tau$ . Results of replica analysis are consistent with the averages obtained from a numerical experiment with 100 samples and  $N = 500$  assets.

estimated by numerical experiments for purposes of comparison. As shown, the results derived using replica analysis are consistent with those estimated using numerical simulation, whereas the results obtained using the standard operations research approach are not. This implies that the portfolio derived by the approach widely used in operations research based on the principle of expected utility maximization, which minimizes the expected investment risk,  $\vec{w}^{\text{OR}}$ , is not always able to minimize the investment risk  $\mathcal{H}(\vec{w}|X)$ . Thus, the operations research approach may not be able to provide the optimal investment strategy desired by investors.

## V. SUMMARY AND FUTURE WORK

In the present paper, we have discussed the minimal investment risk per asset for a portfolio optimization problem with two imposed constraints, a budget constraint and an investment concentration constraint, using replica analysis, which was developed for and improved during interdisciplinary research. Unlike the minimal investment risk per asset and portfolio optimization problem with a budget constraint which has been discussed in previous work, we assessed quantitatively the deviation of the minimal investment risk per asset from the budget constraint only case caused by the inclusion of an investment concentration constraint. Moreover, we

could estimate the typical behavior of minimal investment risk per asset using the Lagrange method of undetermined multipliers, which is an algorithm for finding the optimal portfolio. The results obtained using the proposed method, those obtained by a standard operations research approach, and numerical results were compared. We found that the numerical simulation results were consistent with those of replica analysis. In contrast, the standard operations research approach failed to identify accurately the minimal investment risk of the portfolio optimization problem, since the obtained optimal portfolio only minimizes the expected investment risk, not the investment risk, making it clear that this approach cannot provide investors information about the optimal investment strategy.

For simplicity of discussion, we have assumed, in a similar way to in previous work, that the return rate of each asset is normalized as having mean 0 and variance 1. However, in future work, we need to improve and develop the model in order to be able to treat a more realistic depiction of the investment market. For instance, we need to analyze the portfolio optimization problem in an investment market comprising a risk-free asset and assets of varying risk levels. In addition, as alternative constraints to a budget constraint or an investment concentration constraint, we need to consider, for instance, an expected return constraint for the case that the return rate is not normalized and linear inequality constraints. For such research, since there has been little investigation based on statistical mechanical informatics of the various issues related to portfolio optimization problems, there are several unresolved issues in this researching field.

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